

# Gribov vs BRST

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We investigate the way in which the Gribov problem is manifested in the BRST quantization of simple quantum mechanical models by comparing models with and without a Gribov problem. We show that the hermiticity and nilpotency of the BRST charge together with the Batalin-Vilkovisky theorem yield *non-trivial* supplementary conditions on gauge fixing fermions. If the gauge fixing fermion satisfies the supplementary conditions, the BRST physical states form a space isomorphic to the Dirac space, and the BRST formal path integral does not suffer from the Gribov problem. The conventional gauge fixing fermion, that gives rise to the Faddeev-Popov integral, fails to satisfy the supplementary conditions due to the Gribov problem. Alternatively, enforcing the conventional gauge fixing fermion, these supplementary conditions imply restrictions on the BRST physical states for which the Batalin-Vilkovisky theorem holds. We find that these BRST physical states are *not* isomorphic the Dirac states. This can be interpreted as a violation of the Batalin-Vilkovisky theorem on the space of Dirac states and implies a breakdown of unitarity and a general dependence of physical quantities on the gauge condition.

## I. INTRODUCTION.

The Gribov problem is one of the main obstacles in the non-perturbative quantization of gauge theories [1]. Singer's work [2] has shown that the Gribov problem is deeply rooted in the global structure of the configuration space of a gauge theory and that it can not be easily avoided unless one resorts to rather unconventional gauge fixing conditions with mostly undesirable features.

The way in which the Gribov problem effects the conventional Faddeev-Popov path integral has been investigated by many authors, and several suggestions as to how it can be circumvented have been put forward (see e.g. ref. [3] and references therein; [4] for some recent developments). A question which has received much less attention relates to the effect the Gribov problem has on the BRST quantization. Fujikawa suggested that it may be related to a spontaneous breakdown of the BRST symmetry [5]. On the other hand Govaerts pointed out that it may be connected to a violation of the Batalin-Vilkovisky theorem [6].

Since the discovery of the BRST symmetry of the Faddeev-Popov action, there existed a believe that the path integral for theories with local symmetries can be defined as a path integral for an effective theory with the global BRST symmetry. This is mainly due to the success of the perturbative Yang-Mills theory. It was pointed out by Neuberger [7] that this equivalence breaks down beyond the perturbation theory. The conventional BRST action may give rise to a zero partition function due to the Gribov problem as well as to vanishing expectation values of physical operators.

For some gauge models like topological Yang-Mills theories and 2D topological gravity, the BRST formal path integral can be cured [8]. An extension of this approach to realistic theories is still a problem because the approach heavily relies on some features that are inherent to topological gauge theories, although an interesting proposal has recently been made in [9].

It should be noted that the Hamiltonian path integral allows one to resolve the Gribov problem [10]. The recipe can be tested on soluble gauge models [11]. The Hamiltonian path integral can also be given a form that does not depend on any particular parametrization of the gauge orbit space and, therefore, it is "coordinate-free" [12].

One usually wants to maintain an explicit Lorentz invariance of the path integral. The standard tool to achieve this goal is the BRST formalism. Due to the Gribov problem even the BRST formal path integral is ill-defined and has to be modified. It is important to realise that, regardless of the Gribov problem, the path integral measure of a field theory is ill-defined beyond the perturbation theory (e.g. [13]). Thus it must be regularised before performing

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any formal manipulations that aim at the possible cure of the Gribov disease (see, e.g., a lattice approach in [14]). Since the BRST formalism has a well defined operator interpretation, it seems rather natural to localise the Gribov problem in the BRST operator approach and amend the path integral accordingly. This is the aim of this letter.

Our main concern here will therefore be the construction of the physical subspaces, following the BRST quantization, of models exhibiting the Gribov problem and those not possessing the Gribov problem. From this we extract the distinguishing feature which isolates the effect of the Gribov problem.

We proceed along the conventional lines of BRST quantization and the normal assumptions of quantum mechanics. Thus we quantize in the Hilbert space with conventional inner product and with the assumptions that (1) the BRST operator is hermitian and (2) physical observables are hermitian on the physical subspace [15].

Our approach is somewhat unconventional when we construct the physical subspace. As usual we construct the physical states as the zero ghost number states annihilated by the BRST charge [15]. In addition, however, we also require explicitly that physical transition amplitudes are independent from the gauge fixing fermion, i.e., that the Batalin-Vilkovisky theorem holds [16]. This is done in order to make the tacit assumptions behind the Batalin-Vilkovisky theorem explicit. In so doing, we end up with some supplementary conditions on admissible gauge fixing fermions, which are crucial for the sequel analysis of the Gribov problem. These conditions are model independent and apply to any BRST quantized system. Once these supplementary conditions have been identified, one can take two points of view. One can either consider them as a constructive procedure for obtaining the admissible class of gauge fixing fermions which yields a physical subspace, identified as usual through the cohomology, isomorphic to the Dirac space. Alternatively, given the gauge fixing fermion one can view these as restrictions on the physical subspace on which the Batalin-Vilkovisky theorem holds. This subspace may not be isomorphic to the Dirac space, which would imply that the BRST quantization with this particular gauge fixing fermion is not equivalent to Dirac quantization and hence it should be discarded. Stated differently, the Batalin-Vilkovisky theorem is violated on the full Dirac space for this gauge fixing fermion. This implies a breakdown of unitarity and leads to a dependence of physical quantities on the gauge condition in the full Dirac space.

Here we analyse the Gribov problem for the conventional BRST quantization using a conventional gauge fixing fermion which leads to the Fadeev-Popov path integral. Using this gauge fixing fermion, we find in the presence of the Gribov problem that no globally well defined gauge fixing condition exists which yields a physical subspace isomorphic to the set of Dirac states. In particular such a gauge fixing condition can not be of the class  $C^2$  everywhere. The class of conventional gauge fixing fermions with globally well defined gauge fixing conditions should therefore be considered as invalid. This seems to confirm the conclusions of [6].

The paper is organised as follows: In section 2 we present the two quantum mechanical models we study and discuss their BRST quantization. In section 3 we derive our supplementary conditions on gauge fixing fermions. Section 4 is devoted to the Gribov problem. We construct the physical subspaces associated with the models and show how the Gribov problem manifest itself for the conventional gauge fixing fermion (the latter fails to satisfy the supplementary conditions). In section 5 we present our conclusions and discuss possible ways to circumvent the Gribov problem in the BRST formalism.

## II. THE MODELS

The Lagrangians describing the models we consider are

$$L_1 = \frac{1}{2} (\dot{x} - y T x)^2 - V(x^2), \quad (2.1a)$$

$$L_2 = \frac{1}{2} \dot{x}_1^2 + \frac{1}{2} (\dot{x}_2 - y)^2 - V(x_1), \quad (2.1b)$$

with  $x \in \mathbb{R}^2$ ,  $y \in \mathbb{R}$  and  $T$  the  $2 \times 2$  anti-symmetric matrix with  $T_{12} = 1$ . We shall refer to (2.1a) as model I and (2.1b) as model II. Both models exhibit a gauge symmetry. In the case of model I we have invariance under the transformation  $x \rightarrow e^{\theta T} x$ ,  $y \rightarrow y + \dot{\theta}$  and for model II under  $x_2 \rightarrow x_2 + \theta$ ,  $y \rightarrow y + \dot{\theta}$ . As normal the gauge invariance is associated with the existence of constraints. For both models the primary constraint is  $p_y = 0$  and the secondary constraint is a generator of gauge transformations in the  $x$ -space

$$\sigma_1 = p_x T x = 0, \quad \sigma_2 = p_{x_2} = 0 \quad (2.2)$$

for models I and II, respectively; here  $p_q$  denotes the momenta canonically conjugate to variable  $q$ . The Hamiltonians associated with the models are ( $i = 1, 2$ )

$$H_i^s = \frac{1}{2} p_x^2 + V_i(x) \quad (2.3)$$

with  $V_1$  a function of  $x^2$  only and  $V_2$  a function of  $x_1$  only.

We want to stress that model I has a Gribov problem, but model II not [10]. Consider for example a gauge fixing condition  $\chi(x)$  such that it is continuous and single valued. The equation  $\chi(x) = 0$  determines a line passing through the origin on the  $x$ -plane. Due to the single-valuedness of  $\chi(x)$ , this line intersects each gauge orbit, being a circle centred at the origin, at least twice. For model II it is, however, possible to find gauge conditions with the above properties such that the gauge condition has a unique solution (e.g.  $\chi(x) = x_2 = 0$ ).

To BRST quantize, one extends the classical phase space to include the Lagrange multiplier  $y$  and its canonically conjugate momenta  $p_y$ . One also introduces ghost and anti-ghost degrees of freedom, associated with the two constraints in each model, and their canonically conjugate momenta [15]. The extended phase space is thus  $(x, p_x, y, p_y, \eta, p_\eta, \bar{\eta}, p_{\bar{\eta}})$  where all variables are taken to be real. We shall also use the collective notations  $q^i, \theta^\alpha$  for boson and ghost coordinates, respectively, and  $p_j, \pi_\beta$  for the corresponding momenta.

Quantization proceeds by promoting the canonical variables to linear operators acting in some Hilbert space and satisfying the canonical commutation relations

$$[q^i, p_j] = i \delta_j^i, \quad [\theta^\alpha, \pi_\beta] = \delta_\beta^\alpha, \quad (2.4)$$

where  $[A, B] = AB - (-1)^{\epsilon_A \epsilon_B} BA$  is the graded commutator ( $\epsilon$  is the Grassmann parity,  $\epsilon = 0, 1$  for even and odd variables, respectively). One introduces the ghost number operator [15]

$$N = \eta p_\eta - \bar{\eta} p_{\bar{\eta}}, \quad \epsilon_N = 0, \quad (2.5)$$

and the nilpotent BRST charge for the two models [15]

$$Q_i = p_y p_{\bar{\eta}} + \eta \sigma_i, \quad Q_i^2 = 0, \quad \epsilon_Q = 1. \quad (2.6)$$

Finally one extends the dynamics to the full phase space by introducing the BRST extended Hamiltonian [15]

$$H_i = H_i^s + [Q_i, \rho]. \quad (2.7)$$

Here  $\rho$  is a gauge fixing fermion of the general form, i.e., a generic operator of the ghost number -1,  $[\rho, N] = -\rho$ . Note that  $\eta, p_{\bar{\eta}}$  have ghost number 1, while  $\bar{\eta}, p_\eta$  have ghost number -1. Thus  $Q_i$  have ghost number 1. Any system observable can be extended to the full phase space in this way. One easily verifies that

$$[Q_i, H_i] = [N, H_i] = 0, \quad (2.8)$$

so that the ghost number and BRST charge are conserved.

We adopt the Schrödinger realisation for the operator algebra (2.4), i.e., the Hilbert space is the space of functions  $\psi(x, y, \bar{\eta}, \eta)$  with inner product [15]

$$\langle \psi | \phi \rangle = i \int dx dy d\bar{\eta} d\eta \psi^* \phi \quad (2.9)$$

(note that  $(d\bar{\eta}d\eta)^* = -d\bar{\eta}d\eta$ , therefore the factor  $i$ ). For the moment we leave the boundary conditions on the function  $\psi(x, y, \bar{\eta}, \eta)$  open. We shall address this issue below. The operator algebra (2.4) is then realised by  $q^i \rightarrow q^i$ ,  $p_i \rightarrow -i \partial / \partial q^i$  and for the ghosts  $\theta^\alpha \rightarrow \theta^\alpha$ ,  $\pi_\beta \rightarrow \bar{\partial} / \partial \theta^\beta$ .

The ghost number provides a grading on the Hilbert space and we can write

$$V = V^0 + V^{-1} + V^1, \quad (2.10)$$

where  $V^0$ ,  $V^{-1}$  and  $V^1$  are, respectively, the ghost number 0, -1 and 1 states

$$\psi^0 = A \bar{\eta} \eta + B \in V^0, \quad \psi^{-1} = C \bar{\eta} \in V^{-1}, \quad \psi^1 = D \eta \in V^1. \quad (2.11)$$

As the ghost number operator is anti-hermitian, one easily verifies that [15]

$$\langle \psi^n | \psi^m \rangle = 0 \quad \text{unless} \quad n + m = 0. \quad (2.12)$$

The conventional BRST procedure is to identify the physical subspace with the zero ghost number cohomology of  $Q$ , i.e.,

$$V_{ph} = H^0 = \frac{(\text{Kern } Q)^0}{(\text{Im } Q)^0}. \quad (2.13)$$

The rationale behind this procedure is that  $Q$ -exact states are orthogonal to physical states and have zero norm. Thus they do not contribute when the matrix elements of any physical observables, i.e., an observable that commutes with  $Q$ , are calculated. Hence  $Q$ -exact states should be identified with the null vector, which leads us to the cohomology. The hermiticity of  $Q$  is rather crucial for this procedure. The reason is that in the absence of hermiticity one fails to conclude that for any state  $\psi^0$  such that  $Q\psi^0 = 0$ ,  $\langle\psi^0|Q\phi\rangle = 0$  or that  $\langle Q\phi|Q\psi\rangle = 0$ . Thus, we have to reduce the Hilbert space  $V$  to a subspace  $V_Q$  such that

$$\langle Q\psi|\phi\rangle = \langle\psi|Q\phi\rangle, \quad \langle Q\psi|Q\phi\rangle = 0, \quad (2.14)$$

for any  $\psi, \phi \in V_Q \subset V$ . The first condition ensures the hermiticity of  $Q$ , while the second condition can be regarded as a consistency condition for the first one, i.e.,  $QV_Q \subset V_Q$ , which follows from the nilpotency of  $Q$ .

For a unifying treatment of the two models, let us denote the constraints  $\sigma_j = -i \frac{\partial}{\partial q^j}$ . For model I  $\sigma_1 = -i \frac{\partial}{\partial \phi}$ ,  $\phi \in [-\pi, \pi)$  and for model II  $\sigma_2 = -i \frac{\partial}{\partial x_2}$ ,  $x_2 \in (-\infty, \infty)$ . We also denote  $q_0 = \pi, \infty$  for models I and II, respectively. By making use of the decomposition (2.11) and the orthogonality relation (2.12), one easily verifies that hermiticity of  $Q$  requires

$$\int dx A^* C|_{y=-\infty}^{y=+\infty} = \int d\tilde{x} dy B^* C|_{q=-q_0}^{q=+q_0} = 0, \quad (2.15a)$$

while the second condition in (2.14) yields

$$\int dx (\sigma C_2)^* C_1|_{y=-\infty}^{y=+\infty} = \int d\tilde{x} dy (p_y C_2)^* C_1|_{q=-q_0}^{q=+q_0} = 0. \quad (2.15b)$$

Here  $d\tilde{x}$  denotes the measure after the  $q$  degree of freedom has been integrated out.

To simplify formulas, we adopt the notation  $F^{\pm y}$  for the asymptotic behaviour of the function  $F$  at  $y = \pm\infty$  and similarly  $F^{\pm q}$  for  $F$  at  $q = \pm q_0$ . From the first condition in (2.15b) follows  $(\sigma C)^{\pm y} = 0$  for both models so  $C^{\pm y}$  is gauge invariant. For the first relation in (2.15a) to hold, we have to require that  $A^{\pm y}$  does not have a gauge invariant part, i.e.,  $\int_{-q_0}^{q_0} dq A^{\pm y} = 0$ . Thus we have the conditions

$$\sigma C^{\pm y} = 0, \quad A^{\pm y} = \sigma \bar{a}^{\pm y}, \quad (\bar{a}^{\pm y})^{+q} = (\bar{a}^{\pm y})^{-q}. \quad (2.16a)$$

The second conditions in (2.15b) gives  $p_y C_I^{+q} = p_y C_I^{-q}$  and  $p_y C_{II}^{\pm q} = 0$  for models I and II, respectively. The second condition in (2.15a) implies for model I  $B_I^{+q} = B_I^{-q}$ , while  $\int_{-\infty}^{\infty} dy B^{\pm q} = 0$  for both models. The boundary conditions are therefore

$$p_y C_I^{+q} = p_y C_I^{-q}, \quad p_y C_{II}^{\pm q} = 0, \quad (2.16b)$$

$$B_I^{+q} = B_I^{-q}, \quad B^{\pm q} = p_y \bar{b}^{\pm q}, \quad (\bar{b}^{\pm q})^{+y} = (\bar{b}^{\pm q})^{-y}. \quad (2.16c)$$

The indices I and II refer to models I and II, respectively, while the index is omitted if it applies to both models. All other boundary conditions are still open and will be fixed below.

A point to note is that a sufficient, but not necessary, condition to ensure hermiticity of  $Q$  (see Eqs. (2.6) and (2.9)) is to impose hermiticity of  $p_y$  and  $\sigma$ . We refrain from imposing this stronger condition for two reasons (1)  $p_y$  and  $\sigma$  are  $Q$ -exact and thus not physical and (2) it is easy to see that this condition leads to more stringent boundary conditions on  $A$  and  $B$  than (2.16) so that it yields a smaller physical subspace.

### III. THE PHYSICAL SUBSPACE

Now that we have identified the Hilbert space in which the physical subspace should be sought, i.e., the cohomology problem should be solved, we can proceed to do so. We therefore have to solve the conditions for a physical state [15]

$$N \psi_{ph} = Q \psi_{ph} = 0, \quad (3.1)$$

under the conditions (2.16).

Conventionally the physical subspace,  $V_{ph}$ , is defined as the quotient space obtained after factoring all  $Q$ -exact states from the space (3.1). It is then argued that physical transition amplitudes do not depend on the gauge fixing fermion, i.e., that the Batalin-Vilkovisky theorem holds. This argument is based on the observation that

$$e^{-itH_i} |\psi_{ph}\rangle = e^{-itH_i^s} |\psi_{ph}\rangle + Q|\phi\rangle \quad (3.2)$$

where  $H_i$  and  $H_i^s$  were defined in (2.7) and (2.3), respectively. Since  $Q$ -exact states are orthogonal to physical states and have zero norm, the gauge dependence, which only enters through the  $Q$ -exact part, drops out from the physical transition amplitude. Although this argument is formally correct, it tacitly assumes that the state  $|\phi\rangle$  belongs to the Hilbert space on which  $Q$  is hermitian and nilpotent, i.e.,  $\phi \in V_Q$ . In particular, when applied to models I and II, the boundary conditions (2.16) should hold for this state. Clearly this is by no means guaranteed and depends on the properties of the gauge fixing fermion because  $\phi$  depends on it. One can therefore take the point of view that the gauge fixing fermion has to be restricted to a certain class. Thus we impose this explicitly: The admissible gauge fixing fermion is the one for which the Batalin-Vilkovisky theorem holds and the quotient space  $V_{ph}$  is isomorphic to the set  $V_{Dir}$  of the Dirac states.

Let us now turn to derivation of the supplementary conditions on  $\rho$ . For a physical transition amplitude to be independent of the gauge fixing fermion, we have to require

$$\langle \psi_{ph} | e^{-itH_i} \phi_{ph} \rangle = \langle \psi_{ph} | e^{-itH_i^s} \phi_{ph} \rangle. \quad (3.3)$$

Expanding the exponential on the left hand side of (3.3) we note that a generic term has the form

$$\langle \psi_{ph} | Q [\rho (Q\rho)^{m_1} (H_i^s)^{n_1} (Q\rho)^{m_2} (H_i^s)^{n_2} \dots (Q\rho)^{m_n}] \phi_{ph} \rangle, \quad (3.4)$$

where we have used (3.1) and  $[Q, H_i^s] = 0$ . We note from (3.4) that for (3.3) to hold, we must require

$$\rho\psi_{ph} \in V_Q^{-1} \quad \text{if} \quad \psi_{ph} \in V_{ph} \sim V_{Dir}; \quad (3.5a)$$

$$Q\rho\psi_{ph} \in V_{ph} \quad \text{if} \quad \psi_{ph} \in V_{ph} \sim V_{Dir}; \quad (3.5b)$$

$$H_i^s\psi_{ph} \in V_{ph} \quad \text{if} \quad \psi_{ph} \in V_{ph} \sim V_{Dir}. \quad (3.5c)$$

The first condition can be viewed as a *kinematical* supplementary condition on  $\rho$ . It ensures that in  $V_Q$  matrix elements of any gauge invariant operator, extended to the total BRST Hilbert space according to the rule (2.7), do not depend on the extension. The validity of the chain of equalities  $\langle \psi | [Q, \rho] \phi \rangle = \langle \psi | Q\rho\phi \rangle = \langle Q\psi | \rho\phi \rangle = 0$  for  $Q\phi = Q\psi = 0$  implies that the operator  $\rho$  does not throw states out of  $V_Q$ ,  $\rho V_Q \subset V_Q$ . It is no trivial condition because the function  $C$  is not arbitrary for states belonging to  $V_Q$  (cf. (2.16)).

The second condition is a simple consequence of (2.14) and (3.5a). Clearly, if  $\rho\psi \in V_Q$  for any  $\psi \in V_Q$ , then from (2.14) follows that  $Q\rho\psi \in V_Q$ .

The third condition will be referred to as a *dynamical* supplementary condition since it depends on the system Hamiltonian. Although this condition may seem trivial at first sight because  $[Q, H^s] = 0$ , it is, in fact, highly non-trivial. An important point to note is that this condition is not automatically compatible with the kinematical condition and, in general, gives rise to some further restrictions on the gauge fixing fermion.

We stress the following facts:

1) The conditions (2.14) and (3.5) are model-independent and apply to any BSRT quantized gauge theory. They are by no means guaranteed for a given gauge fixing fermion;

2) If one succeeds to find  $\rho$  satisfying (3.5), the corresponding BRST path integral will not suffer from the Gribov problem at all (if such a problem is present). This follows from the Batalin-Vilkovisky theorem. One can therefore take the point of view that these conditions provide a constructive procedure of finding the class of admissible gauge fixing fermions;

3) Alternatively, if the gauge fermion is fixed, the supplementary conditions we have obtained can be regarded as restrictions on the admissible Hilbert space where the Batalin-Vilkovisky theorem holds and the BRST cohomologies are sought. In the latter case, the Gribov problem may destroy the isomorphism  $V_{ph} \sim V_{Dir}$ . This indeed takes place for the conventional gauge fixing fermion. Thus, due to the Gribov problem, the conventional gauge fixing fermion fails to satisfy the supplementary conditions as we demonstrate in the next section.

#### IV. GRIBOV PROBLEM

Let us take the conventional gauge fixing fermion

$$\rho = -yp_\eta + \chi(x)\bar{\eta}, \quad (4.1)$$

where  $\chi(x)$  is the gauge condition. With this choice one gets the Faddeev-Popov path integral in the gauge  $\dot{y} - \chi(x) = 0$  from the formal BRST path integral by integrating out canonical momenta.

We construct the physical subspace satisfying the conditions (3.1) and (3.5), subject to the boundary conditions (2.16). A general ghost number zero state has the form  $A\bar{\eta}\eta + B$ . The condition  $Q\psi_{ph} = 0$  yields

$$\psi_{ph} = (\sigma a - \sigma F)\bar{\eta}\eta + p_y F, \quad (4.2)$$

where  $p_y a = 0$  and from (2.16)

$$(F^{\pm y})^{+q} - (F^{\pm y})^{-q} = a^{+q} - a^{-q}, \quad (4.3a)$$

$$(F^{\pm q})^{+y} - (F^{\pm q})^{-y} = 0 \quad (4.3b)$$

It is important to note the order of limits here, and to keep in mind that the order can in general not be interchanged, e.g.,  $(F^{+y})^{+q} \neq (F^{+q})^{+y}$  in general.

Before proceeding, we calculate the norm of the states (4.2). Invoking the boundary condition (2.16) and (4.3b) we get

$$\langle \psi_{ph} | \psi_{ph} \rangle = \int dx \{ ((\sigma a)^* F + (\sigma F)^* a - (\sigma F)^* F) |_{y=-\infty}^{y=\infty} \} + \int d\tilde{x} dy (p_y F)^* F |_{-q_0}^{q_0}. \quad (4.4)$$

We first analyse model I. The kinematical supplementary condition (3.5a) gives rise to two conditions on the new  $C$  (see (2.16a) and (2.16b)). Combining these with (2.16c), which yields  $p_y F^{+q} = p_y F^{-q}$ , and assuming  $\chi$  to be single valued result in

$$[-y\sigma^2(a - F) + \sigma\chi p_y F]^{\pm y} = 0, \quad (a - F)^{+q} = (a - F)^{-q} \quad (4.5)$$

From (4.5) follows the asymptotic behaviour  $(p_y F)^{\pm y} = y b_0^{\pm}(x)$  and  $(\sigma F)^{\pm y} = f^{\pm}$ ,  $p_y f^{\pm} = 0$ . These conditions are compatible if  $\sigma b_0^{\pm} = 0$ , i.e.,  $b_0^{\pm}$  is gauge invariant. Now we require that physical states have non-zero norm. First consider the last term in (4.4). From (2.16c), (4.5) and (4.3b) we note that this term vanishes. For an arbitrary  $\sigma a \neq 0$  the first term in (4.4) is finite if  $F^{\pm y} < \infty$  and, hence,  $b_0^{\pm}$  must be zero. The condition (4.5) with  $b_0^{\pm} = 0$  implies that  $F^{\pm y} = \gamma_0^{\pm} + a$ ,  $\sigma\gamma_0^{\pm} = 0$ . Substituting in (4.4) we find that the term in curly brackets also vanishes. Hence the states have zero norm. Thus, for non-zero norm states  $a \equiv 0$ . Note that the states with  $a \neq 0$  can be written as  $Q$ -exact states,  $QC\bar{\eta}$ ,  $C = F - a$ . With  $b_0^{\pm} = 0$  (4.5) ensures (2.16a),  $(\sigma C)^{\pm y} = 0$ .

If  $a = 0$ , the condition  $b_0^{\pm} = 0$  can be dropped. The growing asymptotic of  $F^{\pm y}$  provided by the non vanishing  $b_0^{\pm}$  does not lead to a divergent norm because it is projected out by  $\sigma$  (recall  $\sigma b_0^{\pm} = 0$ ). The non-zero contribution to the integral (4.4) comes from  $(\sigma F)^{\pm y}$  which is required to be finite. The condition (4.5) leads to

$$\sigma(\sigma F^{\pm y} + \chi b_0^{\pm}) = 0, \quad F^{+q} = F^{-q}. \quad (4.6)$$

Decomposing  $\chi = \chi_0 + \sigma\chi'$ ,  $\sigma\chi_0 = 0$  and using (4.3b), we get for the set of physical states (3.1)

$$\psi_{ph} = \{ -\sigma F\bar{\eta}\eta + p_y F : F^{\pm y} = iy^2 b_0^{\pm}/2 - \chi' b_0^{\pm}, \sigma b_0^{\pm} = 0, F^{+q} = F^{-q}, (F^{+q})^{\pm y} = (F^{-q})^{\mp y} \}, \quad (4.7)$$

where the single valuedness of  $\chi'$  was used to note that (4.3a) ( $a = 0$ ) is satisfied. The norm of these states are given by

$$\langle \psi_{ph} | \psi_{ph} \rangle_I = \int dx [(\chi'\sigma\chi') [b_0^{+*} b_0^{+} - b_0^{-*} b_0^{-}]] . \quad (4.8)$$

The Gribov problem manifests itself already on the *kinematical* level. Indeed, the global (in the sense that it is independent of the state vector) term  $\int_{-\pi}^{\pi} d\theta \chi' \sigma \chi'$  factorizes from (4.8). A single valued  $\chi'$  will be periodic in  $\theta$  so that this term vanishes and all physical states have zero norm as a consequence of the kinematical supplementary condition (3.5a). This is, however, easily circumvented since, seeing that this is a global factor, one can still proceed to calculate expectation values of physical observables by simply regularising this factor in an appropriate way. The regularisation dependent global factor  $\int_{-\pi}^{\pi} \chi' \sigma \chi'$  then drops out when calculating expectation values.

Assuming some regularisation of this global factor, we have the important property  $V_{ph} \sim V_{Dir}$ . This is easy to see. Decomposing  $p_y F \equiv B = B_0 + \tilde{B}$  with  $\sigma B_0 = 0$ , we note that  $B_0^{\pm y} = y b_0^{\pm y}$ . Consider  $B' = B'_0 + \tilde{B}'$  such that  $B_0'^{\pm y} = B_0^{\pm y} = y b_0^{\pm y}$ . Then from (4.6)  $F'^{\pm y} = F^{\pm y}$ . Hence  $F' = F + \delta F$  with  $\delta F^{\pm y} = 0$  and  $B' = B + \delta B$  with  $\delta B = ip_y \delta F$ . Therefore we can express the difference between two such states as  $Q i \delta F \bar{\eta}$  where  $i \delta F \bar{\eta} \in V_Q^{-1}$ . We conclude therefore that the states (4.7) mod  $Q$ -exact states are in a one-to-one correspondence with the Dirac states.

Using physical states with zero (unregularised) norm to evaluate the partition function will lead to a vanishing result. Furthermore the expectation values of physical operators will be ambiguous. This fact lies behind the path integral analysis with the similar conclusion [7]. Neuberger has also mentioned that multi-valued gauges may resolve

the problem, which indeed seems to follow from (4.8). One should, however, be careful with such a conclusion. When deriving (4.7), we have assumed the wave functions to be single valued. If a multi-valued gauge is used, the Hamiltonian (2.7) is no longer single-valued so are the wave functions. Therefore the entire analysis, starting from hermiticity of  $Q$ , has to be revised. It is an open question whether hermiticity of  $Q$  can be reconciled with conventional gauge fixing fermions using multi-valued gauges. If not, the use of such gauges would be invalidated.

As was pointed out above, the vanishing of the norm is not really problematic and can be avoided quite easily. There is a much more serious difficulty, which can not be circumvented, that the BRST formalism encounters in the presence of a Gribov problem. This relates to the violation of the dynamical supplementary condition (3.5c), which implies that the Gribov problem invalidates the Batalin-Vilkovisky theorem. The dynamical supplementary condition implies that the state

$$H_1^s \psi_{ph} = -\sigma H_1^s F \bar{\eta} \eta + H_1^s p_y F \equiv -\sigma F' \bar{\eta} \eta + p_y F' \quad (4.9)$$

should belong to (4.7). Clearly  $p_y F'^{\pm y} = y H_1^s b_0^{\pm} \equiv y b_0'^{\pm}$ . Since the gauge invariant part  $\chi_0$  of  $\chi$  is not involved in (4.7) and, hence, will not affect the sequel analysis, we shall assume it to be zero. The crucial condition is  $\sigma F'^{\pm y} = -\chi b_0'^{\pm}$ , which yields  $[\chi, H_1^s] b_0^{\pm} = 0$ . To maintain the isomorphism with the Dirac states, this has to hold for all scalar square integrable functions, i.e., functions of  $r^2 = x_1^2 + x_2^2$  alone. This is only possible if

$$[\chi, H_1^s] = i f(x) \sigma_1 \quad (4.10)$$

with  $f$  real. Since  $\chi$  is real and  $H_1^s$  is hermitian, consistency requires that  $f$  is a function of  $r$  only. Calculating the commutator we have  $[\chi, H_1^s] = (\partial_1 \chi) \partial_1 + (\partial_2 \chi) \partial_2 + \nabla^2 \chi / 2$  from which we conclude

$$\partial_i \chi = f(x) T_{ij} x_j, \quad \nabla^2 \chi = 0. \quad (4.11)$$

From the first equation in (4.11) we deduce  $\nabla^2 \chi = -i \sigma f$  so that the second equation implies  $\sigma f = 0$ , consistent with our earlier observation that  $f$  is a function of  $r$  only. It follows from (4.11) that

$$[\partial_2, \partial_1] \chi = f' r + 2f, \quad (4.12)$$

i.e., in general  $\partial_2 \partial_1 \chi \neq \partial_1 \partial_2 \chi$  almost everywhere so that  $\chi$  is almost nowhere of class  $C^2$ . Demanding equality we solve  $f(r) = r^{-2}$  and  $\chi = -\arctan(x_2/x_1)$ , so that  $\chi$  is everywhere of class  $C^2$  except at the origin. Note also that  $\chi$  is not a single valued function of  $x$ , indeed  $\chi$  is just the angle  $\theta$ . Going to polar coordinates, the laplacian contains  $\sigma_1^2$  so that the commutator with  $\chi$  is proportional to  $\sigma_1$  as required. It is also amusing to note that this gauge is free from the Gribov problem.

We conclude that there is no globally well defined  $\chi$  such that the dynamical supplementary condition (3.5c) holds. Thus for any globally well defined gauge condition  $\chi$  this implies that the set of states for which the Batalin-Vilkovisky theorem holds, does not coincide with the set (4.7) and, therefore, it is *not* isomorphic to the set of Dirac states. Consequently we have to interpret this as a breakdown of the Batalin-Vilkovisky theorem on the space (4.7) for this particular class of gauge fixing conditions. Since this class of gauge fixing fermions leads to the Fadeev-Popov prescription, it signals a similar problem there.

The above considerations eliminates the class of conventional gauge fixing fermions with globally well defined gauge conditions as a valid class. We have seen above that multi-valued gauges are compatible with the dynamical supplementary condition. However, it is not clear that these gauges are compatible with the hermiticity of  $Q$ , so that this class of gauge fixing fermions may also be invalid. To resolve these difficulties one has to seek more general gauge fixing fermions satisfying the kinematical and dynamical supplementary conditions.

Let us return to model II. Conditions (3.5a) and (2.16) now lead to

$$[-y \sigma^2 (a - F) + \sigma \chi p_y F]^{\pm y} = 0, \quad (4.13a)$$

$$[i \sigma (a - F) + y \sigma p_y F + \chi p_y^2 F]^{\pm q} = 0. \quad (4.13b)$$

As before the asymptotic behaviour following from (4.13a) is  $(p_y F)^{\pm y} = y b_0^{\pm}(x)$ ,  $(\sigma F)^{\pm y} = f^{\pm}$ ,  $p_y f^{\pm} = 0$  and  $\sigma b_0^{\pm} = 0$ .

For a finite norm (see (4.4)) we must have for  $\sigma a \neq 0$  that  $F^{\pm y} < \infty$  and  $p_y F^{\pm q} < \infty$ , so that  $b_0^{\pm} = (\sigma p_y F)^{\pm q} = 0$ . As before  $b_0^{\pm} = 0$  implies  $F^{\pm y} = \gamma_0^{\pm} + a$ ,  $\sigma \gamma_0^{\pm} = 0$ . Substituting, the term in curly brackets in the norm (4.4) vanishes as in model I. Applying  $p_y$  to (4.13b) and using  $(\sigma p_y F)^{\pm q} = 0$  gives  $(\chi p_y^3 F)^{\pm q} = 0$ . Thus (assuming  $\chi^{\pm q} \neq 0$ ) we conclude that  $F^{\pm q}$  is at most quadratic in  $y$ . However, since  $F^{\pm y} < \infty$ , only the constant term is allowed so that  $p_y F^{\pm q} = 0$  and the second term in (4.4) also vanishes, yielding a zero norm. As before the resulting state is  $Q$ -exact.

If  $a = 0$  we have for a finite norm  $(\sigma F)^{\pm y} < \infty$  and  $p_y F^{\pm q} < \infty$ . The conditions (4.13) lead to

$$\sigma(\sigma F^{\pm y} + \chi b_0^{\pm y}) = 0, \quad (-i\sigma F + \chi p_y^2 F)^{\pm q} = 0, \quad (\chi p_y^3 F)^{\pm q} = 0. \quad (4.14)$$

Decomposing  $\chi = \chi_0 + \sigma\chi'$ ,  $\sigma\chi_0 = 0$  and using (4.3b) we identify the set of physical states (3.1) as

$$\psi_{ph} = \{-\sigma F \overline{\eta} \eta + p_y F : F^{\pm y} = iy^2 b_0^{\pm}/2 - \chi' b_0^{\pm}, \sigma b_0^{\pm} = 0, (\sigma F)^{\pm q} = (p_y F)^{\pm q} = 0\}, \quad (4.15)$$

where (4.3a) ( $a = 0$ ) implies the condition  $\chi'^{+q} = \chi'^{-q}$  on the gauge fixing function. The rest of the analysis is as for model I. The set (4.15) mod  $Q$ -exact states is in a one-to-one correspondence with the Dirac states and condition (3.5c) is satisfied provided  $[H_2^s, \chi]b_0^{\pm} = 0$ . The main difference now resides in the structure of  $\sigma$ . In model II  $\sigma_2 = p_2$ , and one easily checks that this is satisfied for any function of  $x_1$  if we choose  $\chi = x_2$ , which yields  $\chi' = x_2^2/2$ . Thus a globally well defined gauge fixing function exists such that the Batalin-Vilkovisky theorem holds on the space (4.15) which is isomorphic to the set of Dirac states. Returning to the norm (4.4) we note that it reduces, as in model I, to (4.8) where the global term that factorizes for  $\chi = x_2$  is  $\int_{-\infty}^{\infty} dx_2 x_2^3$ , which yields after regularisation a well defined definite norm.

It is interesting to note that if the potential  $V(r)$  of model I has a well developed minimum at  $r \neq 0$ , and one expands around this minimum, keeping only leading order terms, then model I reduces 'locally' to model II so that a gauge condition like  $\chi = x_2$  can be used 'locally' in model II.

## V. DISCUSSION AND CONCLUSIONS.

We have shown that the hermiticity and nilpotency of the BRST charge together with the Batalin-Vilkovisky theorem yield non-trivial supplementary conditions on gauge fixing fermions. The supplementary conditions apply to a general BRST quantized gauge theory and provide both the kinematical (Hilbert space) and dynamical (S-matrix) equivalence of the BRST scheme to the Dirac formalism. The BRST path integral is not affected by the Gribov problem, provided the gauge fixing fermion satisfies our supplementary condition. These observations were our main goal.

Next we have demonstrated with simple examples of gauge models that the conventional gauge fixing fermion fails to satisfy the supplementary conditions if the Gribov problem is present. As a consequence, the evolution operator (3.2) throws the states out of the physical subspace (4.7)  $V_{ph} \sim V_{Dir}$  to a larger subspace so that the Batalin-Vilkovisky theorem does not hold. This fact can also be interpreted as a breakdown of unitarity of the physical S-matrix.

To circumvent the Gribov problem in the BRST formalism, one can begin to investigate the following possibilities. Firstly, one can try to find a gauge fixing fermion that fulfils the supplementary conditions we have found. A possible candidate is associated with multi-valued gauges, although it is not clear that this is compatible with the hermiticity of  $Q$ . Secondly, one can consider other possible inner products [17] to weaken the supplementary conditions themselves. This will of course lead to unconventional path integrals. More drastically one can consider dropping the condition of a hermitian BRST charge. This will, however, at the very least, require a drastic revision of the BRST quantization procedure. The reason is simply that one loses the property that BRST exact states are orthogonal to physical states and, particularly, that BRST exact states have zero norm. Thus the basic philosophy of the BRST quantization scheme, namely, that BRST exact states have to be identified with the null vector as they have zero norm, is invalidated. Furthermore one cannot argue that BRST exact states will not contribute to physical transition amplitudes.

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